The material in this handout is taken from: Reinhard Diestel, Graph Theory (5th edition), Springer, 2017.

Given a set $S$ and any $k \in \{1, \ldots, |S|\}$, $[S]^k$ is the collection of all $k$-element subsets of $S$.

A graph $G = (V, E)$ has a finite vertex set $V$ and a finite edge set $E \subseteq [V]^2$. We use $i, j, u, v, w, x$ to denote vertices in $V$. The number $|V|$ of vertices is the order of $G$. A graph of order zero is empty, while graphs of order at most 1 are called trivial. An element of $E$ is denoted by $e$ or $(i, j)$. If $(i, j) \in E$, then $i, j$ denote the endpoints of the edge (the order does not matter). A vertex $i$ is incident with an edge $e$ if $e = (i, j)$. Two vertices $i, j$ are adjacent if $(i, j) \in E$. If $E \equiv [V]^2$, then $G$ is the complete graph (or clique) on $n$ vertices, denoted by $K_n$. Note that $G$ has no self-loops $(i, i)$ because $(i, i) \notin [V]^2$. Moreover, there can be at most one edge in $G$ between any two pair of vertices. Such graphs are often called simple.

If $G' = (V', E')$ is such that $V' \subseteq V$ and $E' \subseteq [V']^2 \cap E$, then $G'$ is a subgraph of $G$. If a subgraph $G'$ is such that $E' \equiv [V']^2 \cap E$, then $G'$ is called the subgraph induced by $V'$.

**Degrees.** The neighborhood of a vertex $v$ of $G$ is the set $N(v)$ of vertices that are adjacent to $v$. The degree $d(v)$ of $v$ is the cardinality of $N(v)$. A vertex with degree zero is isolated. The numbers $\delta(G) = \min \{d(v) : v \in V\}$ and $\Delta(G) = \max \{d(v) : v \in V\}$ are the minimum and maximum degree of $G$. If $\delta(G) = \Delta(G) = k$ then $G$ is $k$-regular.

The average degree of $G$ is

$$d(G) = \frac{1}{|V|} \sum_{v \in V} d(v)$$

and we obviously have $\delta(G) \leq d(G) \leq \Delta(G)$. A related quantity is the edge density $\varepsilon(G) = |E|/|V|$. Note that

$$|E| = \frac{1}{2} \sum_{i \in V} d(v) = \frac{1}{2} d(G)|V|$$

implying $\varepsilon(G) = d(G)/2$.

**Fact 1** The number of vertices of odd degree is always even in any graph.

**Proof.** Since $|E|$ is integer and $|E| = \frac{1}{2} \sum_{v \in V} d(v)$, then $\sum_{v \in V} d(v)$ must be even. Therefore, the number of vertices with odd degree must be even. \qed

We already know that the edge density is half the average degree. Now note that the minimum degree can be larger than the edge density. For instance, in $K_2$ we have $\delta(G) = 1$ and $\varepsilon(G) = \frac{1}{2}$.

**Fact 2** Every $G$ with at least one edge has an induced subgraph $H$ such that $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$. 

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**Proof.** Construct a sequence of nested subgraphs $G \equiv G_0, G_1, \ldots$ induced by the vertex sets $V = V_0 \supseteq V_1 \supseteq V_2 \cdots$ as follows. If $V_i$ has a vertex $v_i$ of degree $d(v_i) \leq \varepsilon(G_i)$ then $V_{i+1} \equiv V_i \setminus \{v_i\}$. Otherwise, stop and set $H = G_i$. If $G_{i+1}$ is created, then

$$\varepsilon(G_{i+1}) = \frac{|E_{i+1}|}{|V_{i+1}|} = \frac{|E_i| - d(v_i)}{|V_i|} \geq \frac{|E_i| - \varepsilon(G_i)}{|V_i| - 1} \geq \frac{|E_i|}{|V_i|} - \varepsilon(G_i) = \varepsilon(G_i)$$

The procedure stops before emptying the graph because $\varepsilon(K_1) = 0 < \varepsilon(G)$. When the procedure stops (say at $H \equiv G_k$ for some $k \geq 0$), it must be that $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$, concluding the proof. \(\square\)

**Paths and cycles.** A path in $G = (V, E)$ of length $k \geq 0$ is a subgraph $P_k$ containing $k + 1$ distinct vertices $v_0, \ldots, v_k \in V$ and $k$ edges $e_1, \ldots, e_k \in E$ such that $e_i = (v_{i-1}, v_i)$ for $i = 1, \ldots, k$. If $k = 0$ then $P_0 = K_1$. A cycle $C_k$ in $G$ of length $k \geq 3$ is formed when a path $P_{k-1}$ can be extended in $G$ to include the edge $(v_{k-1}, v_0) \in E$. The length of a shortest cycle in $G$ is the girth $g(G)$, while the length of a longest cycle in $G$ is the circumference. A chord is any edge between two vertices of a cycle which is not itself an edge of the cycle.

If a graph has a large minimum degree, then it contains long paths and cycles.

**Fact 3** Every graph $G$ with $\delta(G) \geq 2$ contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G) + 1$.

**Proof.** Let $v_0, \ldots, v_k$ be the vertices on any longest path $P_k$ in $G$. Then $N(v_k)$ all belong to $P_k$ (otherwise $P_k$ is not the longest path). Therefore, $k \geq d(v_k) \geq \delta(G)$. Now let $v_i$ the vertex of $P_k$ with smallest index $i$ such that $(v_i, v_k) \in E$. Then the vertices $v_i, \ldots, v_k$ form a cycle of length at least $\delta(G) + 1$ because the degree of $v_k$ is at least $\delta(G)$. \(\square\)

The distance $d(i, j)$ between two vertices $i, j$ is the length of the shortest path between them (if no path exists between the two vertices, then the distance is infinite). The diameter $\text{diam}(G)$ of $G$ is the largest distance between any two vertices in $G$ (note that the diameter can be infinite, for example when the graph has an isolated vertex). The radius $\text{rad}(G)$ of a graph $G$ is the smallest distance $d$ such that there exists a vertex whose distance from any other vertex in $G$ is at most $d$. Formally,

$$\text{rad}(G) = \min_{i \in V} \max_{j \in V} d(i, j)$$

Clearly, $\text{rad}(G) \leq \text{diam}(G)$. Also, let $x \in V$ such that $d(x, v) \leq \text{rad}(G)$ for all $v \in V$. Pick any two vertices $u, v \in V$ then $d(u, v) \leq d(u, x) + d(x, v) \leq 2\text{rad}(G)$. This shows that $\text{diam}(G) \leq 2\text{rad}(G)$.

Girth and diameter are related as follows.

**Fact 4** Every graph $G$ containing at least a cycle satisfies $g(G) \leq 2\text{diam}(G) + 1$.

**Proof.** Let $C$ be the shortest cycle in $G$. If $g(G) \geq 2\text{diam}(G) + 2$, then $C$ contains at least $2\text{diam}(G) + 2$ edges. Take any two vertices $x, y$ at opposite extremes of $C$. Then $x, y$ are connected by two paths in $C$, say $P_1$ and $P_2$, whose each length is at least $\text{diam}(G) + 1$. On the other hand, the distance between $x$ and $y$ in $G$ can be at most $\text{diam}(G)$ by definition of diameter. Let $P$ be a path joining $x$ to $y$ in $G$. Note that not all the edges of $P$ can be in $C$ (otherwise, $C$ is not the
shortest cycle in $G$). Then $P$ together with the shortest between $P_1$ and $P_2$ forms a cycle shorter than $C$, and we have a contradiction. \qed

The next result, which we state without proof, shows that the order of $G$ is lower bounded by a function $n_0(d, g)$ of its average degree $d$ and girth $g \geq 3$, where

$$n_0(d, g) = \begin{cases} 
1 + d \sum_{i=0}^{(g-1)/2-1} (d-1)^i & \text{if } g \text{ is odd} \\
2 \sum_{i=0}^{g/2-1} (d-1)^i & \text{if } g \text{ is even.}
\end{cases}$$

**Theorem 5 (Alon, Hoory, and Linial, 2002)** For any graph $G = (V,E)$, $|V| \geq n_0(d, g)$ for all $d = d(G) \geq 2$ and $g = g(G) \geq 3$.

Using the well-known formula

$$\sum_{i=0}^{n-1} a^i = \frac{a^n - 1}{a - 1} \quad a, n \in \mathbb{N} \quad a > 1, n \geq 1$$

we have that $n_0(d, g) = d^{\Theta(g)}$ for $d \geq 2$. Therefore, the above theorem says that $|V| = d^{\Omega(g)}$. Also, using again the same formula,

$$n_0(3, g) = 2 \sum_{i=0}^{g/2-1} 2^i = 2(2^{g/2} - 1) = 2^{g/2} + 2^{g/2} - 2 = 2^{g/2} + 4 - 2 > 2^{g/2} \quad (\text{for } g \geq 3 \text{ even})$$

and

$$n_0(3, g) = 1 + 3 \sum_{i=0}^{(g-1)/2-1} 2^i = 1 + 3(2^{(g-1)/2} - 1) = \frac{3}{\sqrt{2}} 2^{g/2} - 2$$

$$> \left(\frac{3}{\sqrt{2}} - 1\right) 2^{g/2} - 2 + 2^{g/2} > 2^{g/2} \quad (\text{for } g \geq 3 \text{ odd})$$

Therefore, $n_0(3, g) \geq 2^{g/2}$. So when $\delta(G) \geq 3$ we have $|V| \geq 2^{g(G)/2}$, which in turn implies $g(G) < 2 \log_2 |V|$.

**Connectivity.** A non-empty graph $G$ is **connected** if any two of its vertices are linked by a path in $G$. A maximal connected subgraph of $G$ is a **component** of $G$. Any non-empty graph corresponds to a set containing at least one component. $G$ is $k$-**connected** if $|V| > k$ and for all $X \subset V$ with $|X| < k$, the subgraph induced by $V \setminus X$ is connected. Every non-empty graph is 0-connected, and the 1-connected graphs are precisely the non-trivial connected graphs ($K_1$ is not 1-connected because it does not have order 2).

The largest integer $k$ such that $G$ is $k$-connected is the connectivity $\kappa(G)$ of $G$. Thus $\kappa(G) = 0$ if and only if $G$ is disconnected or $G \equiv K_1$, and $\kappa(K_n) = n - 1$ for all $n \geq 1$.

We now relate connectivity to minimum degree and to the existence of a set of edges whose removal disconnects the graph.
**Theorem 6** If $G$ is non-trivial, then $\kappa(G) \leq |F| \leq \delta(G)$ where $F$ is any smallest set of edges whose removal causes the graph to disconnect.

**Proof.** Let $G = (V, E)$ be non-trivial and let $v$ be any vertex with minimum degree $\delta(G)$. Then $|F| \leq \delta(G)$ because $v$ can be disconnected by removing the edges that are incident with $N(v)$. We now show that $\kappa(G) \leq |F|$ by a case analysis. Let $G' = (V, E \setminus F)$.

**Case 1.** $G$ has a vertex $v$ that is not incident with an edge in $F$. Let $C$ be the component of $G'$ that contains $v$ and consider the set $V_C$ of vertices of $C$ that are incident with an edge of $F$. If we remove these vertices, then $v$ is disconnected from the other component of $G$. Hence $\kappa(G) \leq |V_C|$. On the other hand, no edge in $F$ can have both ends in $C$ (otherwise, $F$ is not minimal). Therefore, $|V_C| \leq |F|$. □

**Case 2.** All vertices of $G$ are incident with some edge in $F$. Pick an arbitrary vertex $v$ and let $C$ be the component of $G'$ that contains $v$. Some $u \in N(v)$ are such that $(v, u) \in F$. The others nodes in the neighborhood of $v$ must belong to $C$ and are incident with distinct edges of $F$ (otherwise, $F$ is not minimal). Therefore $d(v) \leq |F|$, which implies $d(v) = |F| = \delta(G)$, because we already know that $|F| \leq \delta(G)$. As removing $N(v)$ disconnects $v$, we conclude $\kappa(G) \leq \delta(G) = |F|$. □

**Trees and forests.** An acyclic graph, one not containing any cycles, is called a forest. A connected forest is called a tree. (Thus, a forest is a graph whose components are trees.) The vertices of degree 1 in a tree are its leaves, the others are its inner vertices. Every non-trivial tree has a leaf—consider, for example, the ends of a longest path. The next theorem (whose proof is left as exercise) characterizes which graphs are trees.

**Theorem 7** The following assertions are equivalent for a graph $T$:

1. $T$ is a tree;
2. Any two distinct vertices $x, y$ of $T$ are linked by a unique path in $T$ (denoted by $xTy$);
3. $T$ is minimally connected, i.e., removing any edge disconnects $T$;
4. $T$ is maximally acyclic, i.e., adding an edge between any two non-adjacent vertices create a cycle.

A spanning tree of a connected graph $G = (V, E)$ is a subgraph $T = (V, E')$ that is a tree (note that $T$ and $G$ share the same vertex set). A common application of the theorem above is to prove that every connected graph contains a spanning tree: take a minimal connected spanning subgraph and use 3, or take a maximal acyclic subgraph and apply 4. Spanning trees can be found in time $O(|E|)$ via breadth-first or depth-first search of $G$.

**Corollary 8** A connected graph with $n$ vertices is a tree if and only if it has $n - 1$ edges.

**Proof.** First, we prove that a tree has $n - 1$ edges. Note that there always exists a permutation $v_1, \ldots, v_n$ of the vertices of a tree so that every $v_i$ with $i \geq 2$ has a unique neighbour in \{\(v_1, \ldots, v_{i-1}\}\}. Induction on $i = 2, \ldots, n$ shows that the subgraph spanned by the first $i$ vertices has $i - 1$ edges. For $i = n$ this proves the claim.

Conversely, let $G$ be any connected graph with $n$ vertices and $n - 1$ edges. Let $G'$ be a spanning
tree in $G$. Since $G'$ has $n - 1$ edges by the first implication, it follows that $G = G'$. □

A tree $T = (V, E)$ with a fixed root $r$ is a **rooted tree**. Writing $x \leq y$ for $x \in rTy$ then defines a partial order on $V$, the **tree-order** associated with $T$ and $r$.

**Bipartite graphs.** Let $r \geq 2$ be an integer. A graph $G = (V, E)$ is called $r$-partite if $V$ admits a partition into $r$ elements such that every edge has its ends in different elements: vertices in the same partition elements must not be adjacent. Instead of 2-partite one usually says bipartite. An $r$-partite graph in which every two vertices from different partition elements are adjacent is called complete. Note that a bipartite graph is not necessarily connected.

Clearly, a bipartite graph cannot contain an odd cycle, a cycle of odd length. We now prove that also the converse is true.

**Fact 9** If a graph does not contain an odd cycle, then it is bipartite.

**Proof.** Let $G = (V, E)$ be a graph without odd cycles. If $G$ is bipartite, then all its components are bipartite or trivial. So we may assume that $G$ is connected. Let $T = (V, E_T)$ be a spanning tree in $G$, pick a root $r$ and denote the associated tree-order on $V$ by $\leq_T$. For each $v \in V$, the unique path $rTv$ has odd or even length. This defines a bipartition of $V$. We show that $G$ is bipartite with this partition. Let $e = (x, y) \in E$. If $e \in E_T$, say with $x \leq_T y$, then $rTy = rTxy$ and so $x$ and $y$ lie in different partition elements. If $e \notin E_T$, then $C_e = (xTy, e)$ is a cycle (because of 4 in the theorem above), and the vertices along $xTy$ alternate between the two partition elements. Since $C_e$ is even by assumption, $x$ and $y$ again lie in different elements. □

**Euler tours.** A walk (resp., a closed walk) is a path (resp., cycle) whose vertices may not be all distinct. A closed walk in a graph is an Euler tour if it traverses every edge of the graph exactly once. A graph is **Eulerian** if it admits an Euler tour.

**Theorem 10 (Euler, 1736)** A connected graph is Eulerian if and only if every vertex has even degree.

**Proof.** The degree condition is clearly necessary: a vertex appearing $k$ times in an Euler tour (or $k + 1$ times, if it is the starting and finishing vertex and as such counted twice) must have degree $2k$. Conversely, we show by induction on $|E|$ that every connected graph $G = (V, E)$ with all degrees even has an Euler tour. The induction starts trivially with $|E| = 0$. Now let $|E| \geq 1$. Since all degrees are even, we can find in $G$ a non-trivial closed walk that contains no edge more than once (see Exercise 4). Let $W$ be such a walk of maximal length, and write $F$ for the set of its edges. If $F = E$, then $W$ is an Euler tour and we are done. Suppose, therefore, that $E' = E \setminus F$ has at least an edge. For every vertex $v \in V$, an even number of the edges $\{(v, u) : u \in N(v)\}$ lies in $F$ (because $W$ is a closed walk containing each edge only once), so the degrees of the subgraph $G' = (V, E')$ are again all even. Since $G$ is connected, $G'$ has an edge $e$ incident with a vertex on $W$. By the induction hypothesis, the component $C$ of $G'$ containing $e$ has an Euler tour. Concatenating this with $W$, we obtain a closed walk in $G$ that contradicts the maximal length of $W$. □

Euler tours can be found in time $O(|E|)$ using Hierholzer’s algorithm.

**Hamilton cycles.** A Hamilton cycle is a cycle that contains all vertices. A graph is Hamiltonian if it contains a Hamilton cycle.
Unlike Euler tours, only sufficient conditions are known for the existence of Hamilton cycles.

**Theorem 11 (Dirac 1952)** \(\textit{Every graph } G \textit{ with } n \geq 3 \textit{ vertices and } \delta(G) \geq n/2 \textit{ is Hamiltonian.}\)

\textbf{Proof.} Let \(G = (V, E)\) be a graph with \(|G| = n \geq 3\) and \(\delta(G) \geq n/2\). Then \(G\) is connected: otherwise, the degree of any vertex in the smallest component \(C\) of \(G\) would be less than \(|C| \leq n/2\). Let \(P_k = v_0, \ldots, v_k\) be a longest path in \(G\). Let us call \(v_i\) the left end of the edge \((v_i, v_{i+1})\), and \(v_{i+1}\) its right end. By the maximality of \(P_k\), each of the \(d(v_0) \geq n/2\) neighbours of \(v_0\) is the right end of an edge of \(P\), and these \(d(v_0)\) edges are distinct. Similarly, at least \(n/2\) edges of \(P\) are such that their left end is adjacent to \(v_k\). Since \(P\) has fewer than \(n\) edges, it has an edge \((v_i, v_{i+1})\) with both properties.

We claim that the cycle \(C = v_0, v_{i+1}, \ldots, v_k, v_{i-1}, \ldots, v_0\) of length \(k+1\) is a Hamilton cycle of \(G\). Indeed, since \(G\) is connected, \(C\) would otherwise have a neighbour not in \(C\) which could be used to extend \(P_k\), violating the maximality of \(P_k\). \(\square\)

The problem of determining whether an Hamiltonian path exists in a graph is \(NP\)-complete.

\textbf{Exercises.} 

1. Show that every 2-connected graph contains a cycle.
2. Show that every connected graph \(G = (V, E)\) contains a path of length at least \(\min\{2\delta(G), |V| - 1\}\)
4. Show that in every connected graph whose each vertex has even degree there exists a non-trivial closed walk that contains no edge more than once.

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Thanks to Ivan Masnari for spotting a mistake in the previous version of this exercise.