The material in this handout is taken from: Luca Trevisan, Lecture Notes on Graph Partitioning, Expanders and Spectral Methods, 2016.

Given a real $n \times n$ matrix $M$, if $Mu = \lambda u$ for some $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^d \setminus \{0\}$, then $u$ is an eigenvector of $M$ with eigenvalue $\lambda$ (we also say that $u$ is an eigenvector of $\lambda$). Note that eigenvectors can be rescaled without changing the equation $Mu = \lambda u$, hence we conventionally assume they have unit length.

Note that $\lambda$ is an eigenvalue for $M$ if and only if there exists $x \neq 0$ such that $(M - \lambda I)x = 0$, where $I$ is the $n \times n$ identity matrix. The equation $(M - \lambda I)x = 0$ holds for $x \neq 0$ if and only if $M - \lambda I$ is singular, which is equivalent to $\det(M - \lambda I) = 0$. Since $\det(M - \lambda I)$ is an $n$-th degree univariate polynomial in $\lambda$, it has exactly $n$ solutions by the fundamental theorem of algebra. This shows that every square matrix has $n$ eigenvalues (not all necessarily distinct). Some of these eigenvalues, however, may correspond to solutions of $\det(M - \lambda I) = 0$ in the complex plane. The next result guarantees that at least one eigenvalue is real when $M$ is symmetric.

Fact 1 (proof omitted) If $M$ is symmetric, then there exists $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^n \setminus \{0\}$ such that $Mu = \lambda u$.

Fact 2 If $M$ is symmetric then any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. Let $x$ be an eigenvector of $\lambda$ and $y$ an eigenvector of $\lambda'$. Since $M$ is symmetric, $(Mx)^\top y = x^\top My$. On the other hand, $(Mx)^\top y = \lambda x^\top y$ and $x^\top My = \lambda' x^\top y$. Since $\lambda \neq \lambda'$, it must by $x^\top y = 0$, which means that $x$ and $y$ are orthogonal. □

Theorem 3 (Spectral Theorem) Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. Then there exists $n$ (not necessarily distinct) real numbers $\lambda_1, \ldots, \lambda_n$ and $n$ orthonormal real vectors $u_1, \ldots, u_n$ such that $u_i$ is an eigenvector of $\lambda_i$.

Proof. The proof is by induction on $n$. If $n = 1$, then $M$ is a scalar. Hence, $Mx = Mx$ shows that any nonzero $x \in \mathbb{R}$ is an eigenvector of $M$ with eigenvalue $M$.

Assume now that the statement holds for $n - 1$. By Fact 1, there exist an eigenvalue $\lambda_1 \in \mathbb{R}$ with eigenvector $x_1 \in \mathbb{R}^n$. Now we claim that $y$ orthogonal to $x_1$ implies $My$ is orthogonal to $x_1$. Indeed, $x_1^\top My = (Mx_1)^\top y = \lambda x_1 y = 0$.

Let $V$ be the $n - 1$-dimensional subspace of $\mathbb{R}^n$ that contains all the vectors orthogonal to $x_1$. We can define a one-to-one linear mapping $B : \mathbb{R}^{n-1} \to V$ by choosing an orthonormal basis $u_1, \ldots, u_{n-1}$ for $V$ and letting $B = [u_1, \ldots, u_{n-1}]$. Also, $BB^\top$ is a projection matrix that projects
vectors from \( \mathbb{R}^n \) to \( V \). In particular, \( BB^T y = y \) for all \( y \in V \). We now apply the inductive hypothesis to the \((n - 1) \times (n - 1)\) matrix \( M' = B^T MB \) and find real eigenvalues \( \lambda_1, \ldots, \lambda_{n-1} \) and orthonormal eigenvectors \( y_1, \ldots, y_{n-1} \). For \( 2 = 1, \ldots, n \) we have \( M'y_i = B^T MB y_i = \lambda_i y_i \). Therefore, \( BB^T MB y_i = \lambda_i B y_i \). Since \( y_i \in \mathbb{R}^{n-1} \) and \( B \) maps \( \mathbb{R}^{n-1} \) to \( V \), \( By_i \) is orthogonal to \( x_1 \) and, by the above claim, \( MB y_i \) is orthogonal to \( x_1 \). Therefore \( BB^T MB y_i = MB y_i \) and so we have \( MB y_i = \lambda_i B y_i \). If we now define \( x_i = By_i \) we have \( Mx_i = \lambda_i x_i \). To finish up, note that by construction \( x_1 \) is orthogonal to \( x_2, \ldots, x_n \). Moreover, for any \( 2 \leq i < j \leq n \), \( x_i^T x_j = (B y_j)^T (B y_j) = y_i^T B^T B y_j = y_i^T y_j = 0 \). Hence we have found \( n \) eigenvalues with \( n \) eigenvectors.

\[ \square \]

**Corollary 4** Let \( M \in \mathbb{R}^{n \times n} \) be a real symmetric matrix. Then

\[ M = U \Lambda U^T = \sum_{i=1}^{n} \lambda_i u_i u_i^T \]

where \( U = [u_1, \ldots, u_n] \) and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \). Here \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) are the real eigenvalues of \( M \) and \( u_1, \ldots, u_n \in \mathbb{R}^n \) are the corresponding eigenvectors.

**Proof.** Note that \( MU = [\lambda_1 u_1, \ldots, \lambda_n u_n] \) because \( Mu_i = \lambda_i u_i \) for each \( i = 1, \ldots, n \). Hence \( MU = UA \) where \( A = \text{diag}(\lambda_1, \ldots, \lambda_n) \). Since \( U \) is orthonormal, \( U^{-1} = U^T \) and \( UU^T = I \). Therefore \( M = MUU^T = U \Lambda U^T \). \( \square \)

**Theorem 5** (Variational characterization of eigenvalues — proof omitted) Let \( M \in \mathbb{R}^{n \times n} \) be a real symmetric matrix, and \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) be its real eigenvalues. For \( k < n \) let \( u_1, \ldots, u_k \) be orthonormal vectors such that \( Mu_i = \lambda_i u_i \) for \( i = 1, \ldots, k \). Then

\[ \lambda_{k+1} = \min_{u \in \mathbb{R}^n \setminus \{0\}} \frac{u^T Mu}{u^T u} \quad \text{for } u \perp \{u_1, \ldots, u_k\} \]

and any minimizer \( u \) is an eigenvector of \( \lambda_{k+1} \).

The ratio in the right-hand side is called Rayleigh quotient. Note that, in particular,

\[ \lambda_1 = \min_{u \in \mathbb{R}^n \setminus \{0\}} \frac{u^T Mu}{u^T u} \]

Also, because \( -M \) has eigenvalues \( -\lambda_n \leq -\lambda_{n-1} \leq \cdots \leq -\lambda_1 \),

\[ -\lambda_n = \min_{u \in \mathbb{R}^n \setminus \{0\}} \frac{u^T (-M) u}{u^T u} = -\max_{u \in \mathbb{R}^n \setminus \{0\}} \frac{u^T Mu}{u^T u} \]

and therefore

\[ \lambda_n = \max_{u \in \mathbb{R}^n \setminus \{0\}} \frac{u^T Mu}{u^T u} \]