The material in this handout is taken from: Luca Trevisan, Lecture Notes on Graph Partitioning, Expanders and Spectral Methods, 2016.

Given a real $n \times n$ matrix $M$, if $Mu = \lambda u$ for some $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^n \setminus \{0\}$, then $u$ is an eigenvector of $M$ with eigenvalue $\lambda$ (we also say that $u$ is an eigenvector of $\lambda$). Note that eigenvectors can be rescaled without changing the equation $Mu = \lambda u$, hence we conventionally assume they have unit length.

Note that $\lambda$ is an eigenvalue for $M$ if and only if there exists $x \neq 0$ such that $(M - \lambda I)x = 0$, where $I$ is the $n \times n$ identity matrix. The equation $(M - \lambda I)x = 0$ holds for $x \neq 0$ if and only if $M - \lambda I$ is singular, which is equivalent to $\det(M - \lambda I) = 0$. Since $\det(M - \lambda I)$ is a $n$-th degree univariate polynomial in $\lambda$, it has exactly $n$ solutions by the fundamental theorem of algebra. This shows that every square matrix has $n$ eigenvalues (not all necessarily distinct). Some of these eigenvalues, however, may correspond to solutions of $\det(M - \lambda I) = 0$ in the complex plane. The next result guarantees that at least one eigenvalue is real when $M$ is symmetric.

Fact 1 (proof omitted) If $M$ is symmetric, then there exists $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^n \setminus \{0\}$ such that $Mu = \lambda u$.

Fact 2 If $M$ is symmetric then any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. Let $x$ be an eigenvector of $\lambda$ and $y$ an eigenvector of $\lambda'$ with $\lambda \neq \lambda'$. Since $M$ is symmetric, $(Mx)^\top y = x^\top My$. On the other hand, $(Mx)^\top y = \lambda x^\top y$ and $x^\top My = \lambda' x^\top y$. Since $\lambda \neq \lambda'$, it must by $x^\top y = 0$, which means that $x$ and $y$ are orthogonal. $\square$

Theorem 3 (Spectral Theorem) Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. Then there exists $n$ (not necessarily distinct) real numbers $\lambda_1, \ldots, \lambda_n$ and $n$ orthonormal real vectors $u_1, \ldots, u_n$ such that $u_i$ is an eigenvector of $\lambda_i$.

Proof. The proof is by induction on $n$. If $n = 1$, then $M$ is a scalar. Hence, any nonzero $x \in \mathbb{R}$ is an eigenvector of $M$ with eigenvalue $M$ because $Mx = Mx$.

Assume now that the statement holds for $n - 1$. By Fact 1, there exist an eigenvalue $\lambda_n \in \mathbb{R}$ with eigenvector $x_n \in \mathbb{R}^n$.

Claim. $y$ orthogonal to $x_n$ implies $My$ is orthogonal to $x_n$.

Indeed, $x_n^\top My = (Mx_n)^\top y = \lambda x_n^\top y = 0$. 

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Now let $V$ be the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ that contains all the vectors orthogonal to $x_n$. Now choose an orthonormal basis $u_1, \ldots, u_{n-1}$ for $V$ and let $B = [u_1, \ldots, u_{n-1}]$. By construction, $B$ maps $\mathbb{R}^{n-1}$ to $V$ and $BB^T$ maps $\mathbb{R}^n$ to $V$. In particular, $BB^T = z$ for all $z \in V$. We now apply the inductive hypothesis to the $(n-1) \times (n-1)$ symmetric matrix $M' = B^TMB$ and find real eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ and orthonormal eigenvectors $y_1, \ldots, y_{n-1} \in \mathbb{R}^{n-1}$. For $i = 1, \ldots, n-1$ we have $M'y_i = B^TMBy_i = \lambda_i y_i$. Therefore, $BB^TMBy_i = \lambda_i y_i$. Since $y_i \in \mathbb{R}^{n-1}$ and $B$ maps $\mathbb{R}^{n-1}$ to $V$, $By_i$ is orthogonal to $x_n$ and, by the above claim, $MBy_i$ is orthogonal to $x_n$. Therefore $\lambda_i By_i = BB^TMBy_i = MBy_i$. If we now define $x_i = By_i$ for $i = 1, \ldots, n-1$, then we have $Mx_i = \lambda_i x_i$. To finish up, note that by construction $x_n$ is orthogonal to $x_1, \ldots, x_{n-1}$. Moreover, for any $1 \leq i < j \leq n-1$, $x_i^T x_j = (By_i)^T (By_j) = y_i^T B^T By_j = y_i^T y_j = 0$. Hence we have found $n$ eigenvalues with $n$ eigenvectors.\[\square\]

**Corollary 4** Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. Then

$$M = U\Lambda U^T = \sum_{i=1}^{n} \lambda_i u_i u_i^T$$

where $U = [u_1, \ldots, u_n]$ and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Here $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the real eigenvalues of $M$ and $u_1, \ldots, u_n \in \mathbb{R}^n$ are the corresponding eigenvectors.

**Proof.** Note that $MU = [\lambda_1 u_1, \ldots, \lambda_n u_n]$ because $Mu_i = \lambda_i u_i$ for each $i = 1, \ldots, n$. Hence $MU = U\Lambda$ where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Since $U$ is orthonormal, $U^{-1} = U^T$ and $UU^T = I$. Therefore $M = MUU^T = U\Lambda U^T$.\[\square\]

**Theorem 5** (Variational characterization of eigenvalues — proof omitted) Let $M \in \mathbb{R}^{n \times n}$ be a real symmetric matrix, and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be its real eigenvalues. For $k < n$ let $u_1, \ldots, u_k$ be orthonormal vectors such that $Mu_i = \lambda_i u_i$ for $i = 1, \ldots, k$. Then

$$\lambda_{k+1} = \min_{u \in \mathbb{R}^n \setminus \{0\}} \max_{u \perp \{u_1, \ldots, u_k\}} \frac{u^T Mu}{u^T u}$$

and any minimizer $u$ is an eigenvector of $\lambda_{k+1}$.

The ratio in the right-hand side of the above equation is called **Rayleigh quotient**. Note that, in particular,

$$\lambda_1 = \min_{u \in \mathbb{R}^n \setminus \{0\}} \frac{u^T Mu}{u^T u}.$$  

Also, because $-M$ has eigenvalues $-\lambda_n \leq -\lambda_{n-1} \leq \cdots \leq -\lambda_1$,

$$-\lambda_n = \min_{u \in \mathbb{R}^n \setminus \{0\}} \frac{u^T (-M) u}{u^T u} = - \max_{u \in \mathbb{R}^n \setminus \{0\}} \frac{u^T Mu}{u^T u}$$

and therefore

$$\lambda_n = \max_{u \in \mathbb{R}^n \setminus \{0\}} \frac{u^T Mu}{u^T u}.$$  

A symmetric matrix $M$ is **positive semidefinite** if $x^T M x \geq 0$ for all $x \in \mathbb{R}^n$.  

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**Fact 6** The eigenvalues of a positive semidefinite matrix are all nonnegative.

**Proof.** As the denominator of the Rayleigh quotient is clearly always positive, Theorem 5 implies that the sign of each eigenvalue is determined by the sign of $x^T M x$. □

We conclude with a different but equally important characterization of eigenvalues.

**Theorem 7 (Courant-Fischer — proof omitted)** Let $M$ be a real symmetric matrix with real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then

$$\lambda_k = \min_{S: \dim(S) = k} \max_{u \in S \setminus \{0\}} \frac{u^T M u}{u^T u} \quad k = 1, \ldots, n$$

where the minimum is over all subspaces $S$ of dimension $k$. 
