The following result will be used below:

**Theorem 0.1 (Robertson-Seymour).** *For every graph $H$ there exists an algorithm that, for every $G$, decides if $H \preceq G$ in time $O(|V(G)|^3)$.*

## 1 Minor-closed graph families

Recall that a graph property is a family of graphs closed under isomorphism. Many important properties are minor-closed, that is, closed under taking minors.

**Definition 1.1.** A graph family $\mathcal{F}$ is *minor-closed* if $G \in \mathcal{F}$ implies $H \in \mathcal{F}$ for every $H \preceq G$.

**Exercise 1.** Decide if the following graph properties are minor-closed: being acyclic, being planar, having maximum degree at most $k$, having diameter at most $k$.

**Theorem 1.2.** $\mathcal{F}$ is minor-closed if and only if $\mathcal{F} = \text{Forb} (\mathcal{H})$ for some $\mathcal{H}$.

**Proof.** Let $\mathcal{F}^c = \{ G : G \notin \mathcal{F} \}$ be the complement of $\mathcal{F}$.

If $\mathcal{F}$ is minor-closed then every $G \in \mathcal{F}$ satisfies $G \not\preceq H$ for all $H \in \mathcal{F}^c$, while every $G \notin \mathcal{F}$ satisfies $G \succeq G \in \mathcal{F}$. Hence $\mathcal{F} = \text{Forb}(\mathcal{F}^c)$, which proves for $\mathcal{H} = \mathcal{F}^c$ proves the claim.

Now let $\mathcal{F} = \text{Forb}(\mathcal{H})$. If $\mathcal{F}$ is not minor-closed, then there is $G \in \mathcal{F}$ such that $G \succeq G'$ for some $G' \in \mathcal{F}^c$, which thus satisfies $G' \succeq H$ for some $H \in \mathcal{H}$. But then by transitivity $G \succeq H$, which implies $G \notin \mathcal{F}$, a contradiction.

Theorem 1.2 says that every minor-closed graph property has an obstruction set and viceversa.

## 2 The Robertson-Seymour theorem

The following result is among the deepest in graph theory:

**Theorem 2.1 (The Robertson-Seymour graph minor theorem).** *In any infinite sequence of graphs $G_0, G_1, \ldots$ there are indices $i < j$ such that $G_i \preceq G_j$.***

Note that the claim Theorem 2.1 does not hold for the subgraph relation $\subseteq$ (why?). To appreciate Theorem 2.1 let us see two of its consequences.
Consequence #1.

**Theorem 2.2.** \( \mathcal{F} \) is minor-closed if and only if it has a finite obstruction set.

**Proof.** The backward direction is trivial. For the forward direction, define:

\[
\mathcal{H}_\mathcal{F} = \{ H \mid H \in \mathcal{F} \text{ and } \nexists H' \in \mathcal{F} : H' \prec H \}
\]

(1)

It is easy to see that \( \mathcal{F} = \text{Forb}(\mathcal{H}_\mathcal{F}) \). Indeed, if \( G \in \mathcal{F} \) then \( G \not\succeq H \) for all \( H \in \mathcal{H}_\mathcal{F} \) since \( \mathcal{H}_\mathcal{F} \subseteq \overline{\mathcal{F}} \); if \( G \not\in \mathcal{F} \) then \( G \succeq H \) for some \( H \in \mathcal{F} \), and by construction \( \mathcal{H}_\mathcal{F} \) contains either \( H \) or a proper minor. Now let \( H_1, H_2, \ldots \), be any enumeration of \( \mathcal{H}_\mathcal{F} \). By Theorem 2.1 \( \mathcal{H}_\mathcal{F} \) must be finite, otherwise \( H_i \prec H_j \) for some \( i < j \), contradicting the definition of \( \mathcal{H}_\mathcal{F} \). \( \square \)

Consequence #2.

**Theorem 2.3.** Every minor-closed graph property can be decided in time \( O(|V(G)|^3) \).

**Proof.** Let \( \mathcal{F} \) be the property. By Theorem 2.2 \( \mathcal{F} \) has a finite obstruction set \( \mathcal{H} \). To decide whether any given \( G \) is in \( \mathcal{F} \), list every \( H \in \mathcal{H} \) and check whether \( H \preceq G \) in time \( O(|V(G)|^3) \) using the algorithm of Theorem 0.1. Since \( \mathcal{H} \) is fixed (i.e., not part of the input) then the running time is polynomial in the input size. This implies, for instance, the existence of a polynomial-time algorithm for planarity testing. Unfortunately, the constants hidden in the running time of the algorithm of Theorem 0.1 make the algorithm impractical.

### 3 Proof of the graph minor theorem for trees

The proof of Theorem 2.1 is nontrivial; it took two decades and several hundred pages. Here, we prove it for the special case of trees:

**Theorem 3.1.** In any infinite sequence of trees there are two trees \( T, T' \) such that \( T \preceq T' \).

#### 3.1 Colorings and monochromatic subsets

A \( k \)-coloring of a set \( A \) is a function \( c : A \to [k] \). For any set \( X \) let \( [X]^h \) be the set of all \( h \)-sized subsets of \( X \). Thus, a \( k \)-coloring of \( [X]^h \) assigns a color to every \( h \)-sized subset of \( X \). Given a \( k \)-coloring of \( [X]^h \), we say \( Y \subseteq X \) is monochromatic if \( c \) is constant over \( [Y]^h \) (for \( k = 2 \) think of a clique with vertex set \( X \) and a \( k \)-coloring \( c \) of the edges; a monochromatic subset is a sub-clique whose edges have all the same color).

**Theorem 3.2.** Let \( c, h \in \mathbb{N} \) and \( X \) an infinite set. If \( [X]^h \) is coloured with \( c \) colors then \( X \) has an infinite monochromatic subset.

**Proof.** We use induction on \( h \). For \( h = 1 \) the claim is trivial. Let then \( h > 1 \) and assume the claim holds for \( h - 1 \). Let \( X_0 = X \), choose any \( x_0 \in X_0 \), and consider \( [X_0 \setminus \{x_0\}]^{h-1} \). We define a coloring of \( [X_0 \setminus \{x_0\}]^{h-1} \) by letting \( c(Z) = c(\{x_0\} \cup Z) \) for every \( Z \in [X_0 \setminus \{x_0\}]^{h-1} \). By inductive hypothesis, there is an infinite \( Y_0 \subseteq X_0 \setminus \{x_0\} \) such that every \( Z \in [Y_0]^{h-1} \) has the
same color; call it \( c_0 \). Clearly \( c(\{x_0\} \cup Z) = c_0 \) for all \( Z \in [Y_0]^{h-1} \), too. Let \( X_1 = Y_0 \), choose any \( x_1 \in X_1 \), and repeat.

We obtain an infinite sequence of sets \( X = X_0 \supseteq X_1 \supseteq \ldots \) and elements \((x_i)_{i \geq 0}\) with colors \((c_i)_{i \geq 0}\). As the colors are finite, there is an infinite set \( Y = \{x_j : j \geq 0\} \) with the same color. By construction, \( c(Z) \) is constant for all \( Z \in [Y]^h \), hence \( Y \) is monochromatic.

3.2 Well-quasi-orderings

A relation \( \preceq \) over a set \( X \) is a quasi-ordering if it is:

- reflexive: \( x \preceq x \) for all \( x \in X \)
- transitive: \( x \preceq y \) and \( y \preceq z \) implies \( x \preceq z \), for all \( x, y, z \in X \)

(Note that the minor relation is a quasi-ordering). If neither \( x \preceq y \) nor \( y \preceq x \), then \( x \) and \( y \) are incomparable. A set of pairwise incomparable elements is an antichain. A sequence \((x_i)_{i \geq 0}\) is decreasing if \( x_i \succ x_{i+1} \) for all \( i \geq 0 \), and is nondecreasing if \( x_i \preceq x_{i+1} \) for all \( i \geq 0 \). Increasing and nonincreasing sequences are defined similarly. A sequence is good if it contains a good pair, that is, a pair of elements \( x_i \preceq x_j \) with \( i < j \); otherwise the sequence is bad. A quasi-ordering \( \preceq \) on \( X \) is a well-quasi-ordering if every infinite sequence \( x_0, x_1, \ldots \) over \( X \) is good.

**Lemma 3.3.** \( X \) is well-quasi-ordered by \( \preceq \) if and only if \( X \) contains neither an infinite antichain nor an infinite decreasing sequence.

**Proof.** For the forward direction, if \( \preceq \) is a well-quasi-ordering then by definition every infinite sequence contains a good pair and therefore cannot be an antichain or a decreasing sequence.

For the backward direction, let \((x_i)_{i \in \mathbb{N}}\) be any infinite sequence over \( X \) and consider the 3-coloring of \([\mathbb{N}]^2\) defined as follows, assuming without loss of generality that \( i < j \):

\[
c(\{i, j\}) = \begin{cases} 
1 & x_i \preceq x_j \\
2 & x_i \succ x_j \\
3 & x_i, x_j \text{ incomparable}
\end{cases}
\]  

By Theorem 3.2, there is an infinite \( Y \subseteq \mathbb{N} \) such that all elements of \([Y]^2\) have the same colour. In other words there is an infinite subsequence of \((x_i)_{i \in \mathbb{N}}\) that is either increasing, or decreasing, or an antichain. But the last two possibilities are ruled out by hypothesis. Hence \((x_i)_{i \geq N}\) contains an infinite nondecreasing sequence.

The proof above actually shows:

**Corollary 3.4.** \( X \) is well-quasi-ordered by \( \preceq \) if and only if every infinite sequence in \( X \) has an infinite nondecreasing subsequence.

3.3 Well-quasi-orderings of finite subsets

For any set \( X \) we denote by \([X]^{<\omega}\) the set of all finite subsets of \( X \). Every quasi-order \( \preceq \) over \( X \) can be extended to \([X]^{<\omega}\): for every \( A, B \in [X]^{<\omega} \) let \( A \preceq B \) if and only if there is an injection \( f : A \to B \) such that \( a \preceq f(a) \) for all \( a \in A \). It is easy to see that \( \preceq \) is a quasi-order on \([X]^{<\omega}\).
Lemma 3.5. If $X$ is well-quasi-ordered by $\preceq$ then so is $[X]^{<\omega}$.

Proof. Suppose $X$ is well-quasi-ordered by $\preceq$ but $[X]^{<\omega}$ is not. Thus $[X]^{<\omega}$ contains an infinite sequence that is bad. We construct a bad infinite sequence $(A_i)_{i \geq 0}$ that shows that $X$ is not well-quasi-ordered by $\preceq$, a contradiction. Let $A_0 \in [X]^{<\omega}$ be the smallest nonempty set such that there exists a bad infinite sequence in $[X]^{<\omega}$ starting with $A_0$. Now, for every $i = 0, 1, \ldots$, we choose $A_{i+1} \in [X]^{<\omega}$ of minimum cardinality such that there is a bad sequence in $[X]^{<\omega}$ starting with $A_0, \ldots, A_{i+1}$. The sequence $(A_i)_{i \geq 0}$ thus obtained is clearly a bad sequence.

For every $i \geq 0$ choose an arbitrary $a_i \in A_i$. By Corollary 3.4 the sequence $(a_i)_{i \geq 0}$ has an infinite nondecreasing subsequence $(a_{i_j})_{j \geq 0}$. For every $j \geq 0$ define $B_{i_j} = A_{i_j} \setminus \{a_{i_j}\}$, and consider the sequence:

$$S = A_0, \ldots, A_{i_0-1}, B_{i_0}, B_{i_1}, \ldots \quad (3)$$

This sequence $S$ must be good, because if it was bad, then after choosing $A_0, \ldots, A_{i_0-1}$ we should have chosen $B_{i_0}$ instead of $A_{i_0}$. Hence $S$ contains a good pair. We claim that this implies $(A_i)_{i \geq 0}$ being good, a contradiction.

Choose any good pair in $S$. If the pair is in the form $A_i, A_j$ then this implies directly that $(A_i)_{i \geq 0}$ is good. If the pair is in the form $A_i, B_j$ then observe that $B_j \preceq A_j$, hence (by transitivity) $A_i \preceq A_j$, so $A_i, A_j$ is again good. If the pair is in the form $B_i, B_j$ then since $A_i = B_i \cup \{a_i\}$ and $A_j = B_j \cup \{a_j\}$, and since $a_i \preceq a_j$, then once again $A_i \preceq A_j$. Therefore in any case $(A_i)_{i \geq 0}$ is good, which is absurd since it was bad by construction. \qed

3.4 The proof

We can now prove the graph minor theorem for trees.

Theorem 3.6 (Kruskal, 1960). The set of finite trees is well-quasi-ordered by the minor relation.

Proof. The proof actually gives a stronger claim: it holds for rooted trees under the following relation $\preceq$, which is a stronger version of the minor one. Given a tree $T$ with root $t$, the tree-order $\preceq$ over $V(T)$ is such that $x \preceq y$ iff $x$ lies on the path $T(r, y)$ between $r$ and $y$. Given two trees $T, T'$ with roots $r, r'$, let $T \preceq T'$ iff there is an isomorphism $\varphi$ from a subdivision of $T$ to a subtree $T'' \subseteq T'$ that preserves the tree order, i.e., such that if $x \preceq y$ then $\varphi(x) \preceq \varphi(y)$. It is not hard to see that $\preceq$ is a quasi-ordering over the family of rooted trees.

Now suppose by contradiction that the claim was false. As in the proof of Lemma 3.5 construct a bad infinite sequence $(T_i)_{i \geq 0}$ of rooted trees by letting $T_{i+1}$ be any smallest tree (i.e. with the fewest vertices) that extends $T_0, \ldots, T_i$. For every $i \geq 0$ let $r_i$ be the root of $T_i$ and let $A_i$ be the set of rooted trees in $T_i \setminus r_i$ (the roots are the neighbors of $r_i$). We prove that $(A_i)_{i \geq 0}$ contains a pair $A_i, A_j$ with $i < j$ such that for every $T \in A_i$ there is a distinct $T' \in A_j$ satisfying $T \preceq T'$. It is then easy to see that $T_i \preceq T_j$. Hence $(T_i)_{i \geq 0}$ contains a good pair, which is absurd.

Let $A = \bigcup_{i \geq 0} A_i$. We prove that $A$ is well-quasi-ordered. By Lemma 3.3 this implies that $[A]^{<\omega}$ is well-quasi-ordered too, and therefore $(A_i)_{i \geq 0}$, which is an infinite subsequence over $[A]^{<\omega}$, contains a good pair. Let $(T^k)_{k \geq 0}$ be any infinite sequence in $A$. For every $k \geq 0$ choose $n(k)$ such that $T^k \in A_{n(k)}$, and let $k^* = \arg \min_{k \geq 0} n(k)$. Look at the sequence:

$$S = T_0, \ldots, T_{n(k^*)-1}, T^{k^*}, T^{k^*+1}, \ldots \quad (4)$$
Note that $S$ is good: if it was bad, then in the construction of $(T_i)_{i \geq 0}$ we would have chosen $T^k$ instead of $T_{n(k)}$, as $|V(T^k)| < |V(T_{n(k)})|$. The same arguments of the proof of Lemma 3.3 show that any good pair in $S$ has the form $T^k, T^{k_2}$, and thus is a good pair in $(T^k)_{k \geq 0}$, as claimed. □