The material in this handout is mostly taken from: Luca Trevisan, Lecture Notes on Graph Partitioning, Expanders and Spectral Methods, 2016.

Intuitively, a graph is clusterable if its vertices can be partitioned (in a non-trivial way) so that the number of edges across the elements of the partition is small. A key notion is therefore that of cut between disjoint subsets of vertices. We study the clusterability of a graph through the algebraic properties of its adjacency matrix.

Given two disjoints subsets $S, T$ of vertices of a graph $G = (V, E)$, let $E(S, T)$ be the set of edges having one endpoint in $S$ and one endpoint in $T$. Also, let $\neg S = V \setminus S$.

A cut is any partition $(S, \neg S)$ such that $S \not= V$ and $S \not= \emptyset$. The volume $\text{vol}(S)$ of $S \subseteq V$ is the number of edges incident with a node in $S$. The conductance of $S \subseteq V$ is defined by

$$\phi(S) = \frac{|E(S, \neg S)|}{\min \{\text{vol}(S), \text{vol}(\neg S)\}}$$

If $\text{vol}(S) \leq \text{vol}(\neg S)$, this is the fraction of edges in the cut $(S, \neg S)$ among those incident on $S$. If a graph is clusterable, then there exists a partition whose each element $S$ has a small conductance. Finally, the conductance of a graph is

$$\phi(G) = \min_{S : (S, \neg S) \text{ is a cut}} \phi(S).$$

The sparsity of a cut $(S, \neg S)$ is

$$\sigma(S) = \frac{|E(S, \neg S)|}{|S| \cdot |\neg S|}$$

This the fraction of edges in the cut among all potential edges between the two subset of vertices. The sparsity of a graph is

$$\sigma(G) = \min_{S : (S, \neg S) \text{ is a cut}} \sigma(S)$$

In what follows, we focus on $d$-regular graphs for simplicity, where $\text{vol}(S) = d|S|$ and so

$$\min \{\text{vol}(S), \text{vol}(\neg S)\} = d \min\{|S|, |\neg S|\}$$

As $\min\{\alpha, 1 - \alpha\} \leq 2\alpha(1 - \alpha)$ for all $\alpha \in [0, 1]$, we have

$$\min \{|S|, |\neg S|\} \leq \frac{2}{n} |S| \cdot |\neg S|$$

Therefore, $\sigma(S) \leq 2(d/n)\phi(S)$ for all cuts $(S, \neg S)$. We now study the relationships between conductance and algebraic properties of the adjacency matrix.
Laplacian matrix. The Laplacian matrix of a $d$-regular graph $G = (V, E)$ is the symmetric matrix $L = I - \frac{1}{d}A$, where $A$ is the adjacency matrix with entries $A_{i,j} = \mathbb{1}\{i, j \in E\}$. For any $x \in \mathbb{R}^n$ we have that

$$x^T L x = \sum_{i \in V} x_i^2 - \frac{1}{d} \sum_{i \in V} \sum_{j \in V} A_{i,j} x_i x_j$$

$$= \frac{1}{d} \sum_{i \in V} \sum_{j : (i,j) \in E} x_i^2 - \frac{1}{d} \sum_{i \in V} \sum_{j : (i,j) \in E} x_i x_j$$

$$= \frac{1}{d} \sum_{i \in V} \sum_{j : (i,j) \in E} (x_i^2 - x_i x_j)$$

$$= \frac{1}{d} \sum_{(i,j) \in E} (x_i^2 + x_j^2 - 2x_i x_j)$$

$$= \frac{1}{d} \sum_{(i,j) \in E} (x_i - x_j)^2 \geq 0$$

Therefore, the Laplacian matrix is positive semidefinite. Since the rows and columns of $L$ sum to zero (verify that),

$$\lambda_1 = \min_{u \in \mathbb{R}^n \setminus \{0\}} \frac{u^T L u}{u^T u} = 0$$

where the minimum is attained by $u = 1$, where we write $1 = (1, \ldots, 1)$. Hence, $u_1 = \frac{1}{\sqrt{n}} 1$ is the eigenvector of $\lambda_1$, while the remaining eigenvalues of $L$ are all nonnegative because $L$ is positive semidefinite. Note also that any other eigenvector $u_i$ of $L$ with $i > 1$ is such that $u_i^T u_1 = 0$. This helps us characterize $\lambda_2$,

$$\lambda_2 = \min_{u \in \mathbb{R}^n \setminus \{0\}} \frac{u^T L u}{u^T u} = \min_{u \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{(i,j) \in E} (u_i - u_j)^2}{d \sum_{i \in V} u_i^2}.$$ 

If $G = (V, E)$ has two connected components $X, Y \subset V$, then we can choose $u \in \mathbb{R}^n$ such that $u_i = 1/|X|$ for all $i \in X$ and $u_j = -1/|Y|$ for all $j \in Y$. This ensures that $u^T 1 = 0$. Moreover, $(i,j) \in E$ if and only if $(u_i - u_j)^2 = 0$. So $u^T L u = 0$ and therefore $u/\|u\|$ is an eigenvector with eigenvalue $\lambda_2 = 0$. More generally, it can be proven that $\lambda_k = 0$ if and only if $G$ has $k$ connected components. We now look at the largest eigenvalue,

$$\lambda_n = \max_{u \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{(i,j) \in E} (u_i - u_j)^2}{d \sum_{i \in V} u_i^2}$$

$$= \max_{u \in \mathbb{R}^n \setminus \{0\}} \frac{d \sum_{i \in V} u_i^2 - \sum_{(i,j) \in E} 2u_i u_j}{d \sum_{i \in V} u_i^2}$$

$$= \max_{u \in \mathbb{R}^n \setminus \{0\}} \frac{2d \sum_{i \in V} u_i^2 - d \sum_{i \in V} u_i^2 - \sum_{(i,j) \in E} 2u_i u_j}{d \sum_{i \in V} u_i^2}$$

$$= \max_{u \in \mathbb{R}^n \setminus \{0\}} \frac{2d \sum_{i \in V} u_i^2 - \sum_{(i,j) \in E} (u_i + u_j)^2}{d \sum_{i \in V} u_i^2}$$

$$= 2 - \min_{u \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{(i,j) \in E} (u_i + u_j)^2}{d \sum_{i \in V} u_i^2}.$$
Therefore, we have that

\[ \frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2} \]

The second inequality is proven via an efficient algorithm that finds a cut \((S_F, -S_F)\) such that \(\phi(S_F) \leq \sqrt{2\lambda_2}\). Together with the first inequality, this implies that \(\phi(S_F) \leq \sqrt{2\phi(G)}\), which shows how we can efficiently approximate conductance. We begin by proving the first inequality. From now on we write \(\sum_{i=1}^{n} u_i^2\) instead of \(\sum_{i \in V} u_i^2\).

**Lemma 1** For any connected and \(d\)-regular graph \(G\), \(\lambda_2 \leq 2\phi(G)\).

**Proof.** We start noticing that, for any \(\mathbf{u} \in \mathbb{R}^d\) such that \(\mathbf{u}^\top \mathbf{1} = 0\),

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} (u_i - u_j)^2 = 2n \sum_{i=1}^{n} u_i^2 - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} u_i u_j = 2n \sum_{i=1}^{n} u_i^2 - 2 \left( \sum_{i=1}^{n} u_i \right)^2 = 2n \sum_{i=1}^{n} u_i^2 \tag{1}
\]

Therefore, we have that

\[
\lambda_2 = \min_{\mathbf{u} \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{(i,j) \in E} (u_i - u_j)^2}{d \sum_{i=1}^{n} u_i^2} = \min_{\mathbf{u} \in \mathbb{R}^n \setminus \{0\}} \frac{\sum_{(i,j) \in E} (u_i - u_j)^2}{2n \sum_{i=1}^{n} \sum_{j=1}^{n} (u_i - u_j)^2} = \min_{\mathbf{u} \in \mathbb{R}^n \setminus \{0,1\}} \frac{\sum_{(i,j) \in E} (u_i - u_j)^2}{2n \sum_{i=1}^{n} \sum_{j=1}^{n} (u_i - u_j)^2} \tag{2}
\]

To understand the last equality: if \(\mathbf{u} \in \mathbb{R}_n \setminus \{0\}\) and \(\mathbf{u}^\top \mathbf{1} = 0\), then \(\mathbf{u} \neq \mathbf{1}\) and (3) is not larger than (2). Vice versa, if \(\mathbf{u} \notin \{0,1\}\), then \(\mathbf{u}'\) defined by \(u'_i = u_i - \frac{1}{n} \sum_j u_j\) satisfies \(\mathbf{u}' \neq \mathbf{0}\) and \((\mathbf{u}')^\top \mathbf{1} = 0\). Hence, the value of (2) is not larger than (3) because the shift by \(\frac{1}{n} \sum_j u_j\) cancels out in the numerator and the denominator of the objective function.

For any \(S \subseteq V\), let \(\mathbf{u} \in \{0,1\}^n\) be the incidence vector of the set \(S\), that is \(u_i = \mathbb{I}\{i \in S\}\) for \(i = 1, \ldots, n\). Then \(|E(S, \neg S)| = \sum_{(i,j) \in E} (u_i - u_j)^2\). Also, using \(u_i = u_i^2\) for all \(i\),

\[
|S| \ |\neg S| = \left( \sum_{i=1}^{n} u_i^2 \right) \left( n - \sum_{j=1}^{n} u_j^2 \right) = n \sum_{i=1}^{n} u_i^2 - \sum_{i=1}^{n} \sum_{j=1}^{n} u_i u_j = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (u_i - u_j)^2
\]

Therefore,

\[
\sigma(G) = \min_{S \subseteq V : S \neq \emptyset} \frac{|E(S, \neg S)|}{|S| \ |\neg S|} = \min_{\mathbf{u} \in \{0,1\}^n \setminus \{0,1\}} \frac{\sum_{(i,j) \in E} (u_i - u_j)^2}{\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (u_i - u_j)^2}
\]
which implies \( \frac{d}{n} \lambda_2 = \sigma(G) \). Since \( \sigma(G) \leq \frac{2d}{n} \phi(G) \), the proof is concluded.

The proof of the second inequality of Cheeger is based on the analysis of Fiedler’s algorithm, the simplest algorithm for spectral clustering. The algorithm finds a cut of small conductance by looking at the \( n - 1 \) cuts induced by the ranked components of the input vector \( x \). As we see in the analysis, the algorithm works well when \( x \) is the eigenvector of \( \lambda_2 \).

Algorithm 1 (Fiedler)

Input: Graph \( G = (V, E) \), vector \( x \in \mathbb{R}^n \setminus \{0\} \).

1. Sort \( V \) according to the components of \( x \) and let \( v_1 \leq \cdots \leq v_n \) be the vertices of \( V \) after sorting
2. Find \( k \in \{1, \ldots, n-1\} \) minimizing the conductance \( \phi(\{v_1, \ldots, v_k\}) \)

Output: \( \{v_1, \ldots, v_k\} \)

Note that Fiedler’s algorithm can be implemented in time \( O(|E| + |V| \ln |V|) \), because it takes time \( O(|V| \ln |V|) \) to sort the vertices, and the cut of minimal expansion that respects the sorted order can be found in time \( O(|E|) \).

We move on to the analysis of the algorithm, which gives us the second inequality of Cheeger as an immediate consequence. Let

\[
R_L(x) = \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{d \sum_{i=1}^{n} x_i^2}
\]

be the Rayleigh quotient for \( L \) evaluated at \( x \in \mathbb{R}^n \), and recall that

\[
\lambda_2 = \min_{x \in \mathbb{R}^n \setminus \{0\}} \frac{R_L(x)}{x^\top 1 = 0}
\]

We now prove the following result, which implies \( \phi(G) \leq \sqrt{2\lambda_2} \).

**Theorem 2** Let \( x \in \mathbb{R}^n \setminus \{0\} \) be such that \( x^\top 1 = 0 \), and let \( S_F \subset V \) be the cut found by Fiedler’s algorithm with input \( x \). Then \( \phi(S_F) \leq \sqrt{2R_L(x)} \).

Indeed, when the input \( x \) is the eigenvector of \( \lambda_2 \) we get that

\[
\phi(G) \leq \phi(S_F) \leq \sqrt{2\lambda_2}
\]

In order to prove Theorem 2, we need to prove two auxiliary lemmas first.

**Lemma 3** Let \( x \in \mathbb{R}^n \setminus \{0\} \) be such that \( x^\top 1 = 0 \). Then there exists a nonnegative vector \( y \) such that \( R_L(y) \leq R_L(x) \). Furthermore, for every \( 0 < t \leq \max_{v \in V} y_v \), the cut

\[
(\{v \in V : y_v \geq t\}, \{v \in V : y_v < t\})
\]

is one of the cuts considered in line 2 of Fiedler’s algorithm on input \( x \).
PROOF. Let $m$ be the median value of the entries of $x$. Let $x^+, x^-$ have components $x^+_v = [x_v - m]_+$ and $x^-_v = [m - x_v]_+$, where $[z]_+ = z \mathbb{1}(z > 0)$. Note that $x^+, x^-$ are both nonnegative. Now, for every $t > 0$,

$$\{ v \in V : x^+_v \geq t \} = \{ v \in V : [x_v - m]_+ \geq t \} = \{ v \in V : x_v \geq m + t \}$$

is one of the cuts considered by Fiedler’s algorithm on input $x$. Similarly, for every $t > 0$,

$$\{ v \in V : x^-_v \geq t \} = \{ v \in V : [m - x_v]_+ \geq t \} = \{ v \in V : x_v \leq m - t \}$$

is also one of the cuts considered by Fiedler’s algorithm on input $x$. It remains to show that $R_L(y) \leq R_L(x)$ for some nonnegative $y \in \mathbb{R}^n$. We set

$$y = \arg\min_{z \in \{x^+, x^-\}} R_L(z)$$

Let $x' = x - m 1 = x^+ - x^-$ and observe that, for every constant $c$, $R_L(x + c 1) \leq R_L(x)$. Indeed, the numerator of $R_L(x + c 1)$ and the numerator of $R_L(x)$ are the same. Moreover, the denominator of $R_L(x + c 1)$ is $\|x + c 1\|^2 = \|x\|^2 + \|c 1\|^2 \geq \|x\|^2$. Therefore $R_L(x') \leq R_L(x)$ and we are left to show that $R_L(y) \leq R_L(x')$. To this end we write

$$R_L(y) = \min \{ R_L(x^+), R_L(x^-) \}$$

$$\leq \frac{\| x^+ \|^2 R_L(x^+) + \| x^- \|^2 R_L(x^-)}{\| x^+ \|^2 + \| x^- \|^2} \quad \text{(using } \min \{a, b\} \leq \alpha a + (1 - \alpha) b)$$

$$= \frac{\sum_{(i,j) \in E} (x^+_i - x^+_j)^2 + \sum_{(i,j) \in E} (x^-_i - x^-_j)^2}{\| x^+ \|^2 + \| x^- \|^2}$$

$$\leq \frac{\sum_{(i,j) \in E} ((x^+_i - x^+_j) - (x^-_i - x^-_j))^2}{\| x^+ \|^2 + \| x^- \|^2} \quad \text{(this is shown below)}$$

$$= \frac{\sum_{(i,j) \in E} (x'_i - x'_j)^2}{\| x' \|^2} \quad \text{(using } x' = x^+ + x^- \text{ and } (x^+)^\top x^- = 0)$$

$$= R_L(x')$$

To finish the proof, we need to verify that for each $(i, j) \in E$,

$$(x^+_i - x^+_j)^2 + (x^-_i - x^-_j)^2 \leq \left( (x^+_i - x^+_j) - (x^-_i - x^-_j) \right)^2 \quad (4)$$

By computing the square on the right-hand side, the two squares on the left-hand side cancel out with the corresponding squares on the right-hand side. Hence proving (4) is equivalent to proving

$$(x^+_i - x^+_j)(x^-_i - x^-_j) \leq 0 \quad \iff \quad x^+_i x^-_i - x^+_i x^-_j - x^-_j x^-_i + x^+_j x^-_j \leq 0$$

The proof is concluded by observing that $x^+_i x^-_i = x^+_j x^-_j = 0$ by definition, whereas $x^+_i x^-_j \geq 0$ and $x^+_j x^-_i \geq 0$ holds because all the fours factors are nonnegative by definition. \qed

The following observation is used in the proof of the next lemma.
Fact 4 For all random variables \( X, Y \) such that \( Y > 0 \) and \( \mathbb{E}[X], \mathbb{E}[Y] < \infty \),
\[
\mathbb{P}\left( \frac{X}{Y} \leq \frac{\mathbb{E}[X]}{\mathbb{E}[Y]} \right) > 0
\]

Proof. Let \( r = \mathbb{E}[X]/\mathbb{E}[Y] \). Because of linearity of expectation, \( \mathbb{E}[X - rY] = 0 \). Since the expected value is zero, the random variable \( X - rY \) must be nonpositive with probability bigger than zero, \( \mathbb{P}(X - rY \leq 0) > 0 \). Dividing both sides of \( X - rY \leq 0 \) by \( Y > 0 \), we get the desired result. \( \square \)

We are now ready to prove the second auxiliary lemma. Define the expansion of a set \( S \subset V \) by
\[
\text{xpn}(S) = \frac{|E(S, -S)|}{\text{vol}(S)} = \frac{|E(S_t, -S_t)|}{d|S_t|} \quad \text{(for regular graphs)}
\]

Note that, for regular graphs, \( \text{xpn}(S) = \phi(S) \) when \( |S| \leq |\neg S| \).

Lemma 5 For all nonnegative vectors \( y \in \mathbb{R}^n \) there exists \( 0 < t \leq \max_v y_v \) such that
\[
\text{xpn}(S_t) \leq \sqrt{2R_L(y)}
\]
where \( S_t = \{v \in V : y_v \geq t\} \).

Proof. Since rescaling does not affect the Rayleigh quotient, we may assume \( \max_v y_v = 1 \). The proof uses the probabilistic method. Let \( T \) be a random variable such that \( \mathbb{P}(T \leq \sqrt{a}) = a \), which means that \( T^2 \) is uniformly distributed in \([0, 1]\) and \( \mathbb{P}(T \leq 0) = 0 \), which implies \( T > 0 \) with probability 1. Because \( S_t \) is nonempty for all \( t \in (0, 1] \), we can write
\[
\text{xpn}(S_T) = \frac{|E(S_T, -S_T)|}{d|S_T|} \leq \frac{\mathbb{E}[|E(S_T, -S_T)|]}{d\mathbb{E}[|S_T|]} \quad \text{(with probability > 0, by Fact 4)}
\]

This implies that there exists some \( t \in (0, 1] \) such that the above holds. To conclude the proof, we show that
\[
\frac{\mathbb{E}[|E(S_T, -S_T)|]}{d\mathbb{E}[|S_T|]} \leq \sqrt{2R_L(y)}
\]
We start to bound the denominator. Using that \( T \) is uniformly distributed in \([0, 1]\),
\[
\mathbb{E}[|S_T|] = \sum_{i=1}^{n} \mathbb{P}(i \in S_T) = \sum_{i=1}^{n} \mathbb{P}(T \leq y_i) = \sum_{i=1}^{n} y_i^2
\]

(5)

Now pick any \((i, j) \in E\) and assume \( y_j \leq y_i \). Then
\[
\mathbb{P}(i \in S_T, j \in -S_T) = \mathbb{P}(y_j < T \leq y_i) = \left( \mathbb{P}(T \leq y_i) - \mathbb{P}(T \leq y_j) \right) = y_i^2 - y_j^2
\]

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Therefore,
\[ \mathbb{E}\left[ |E(S_T, \neg S_T)|\right] = \sum_{(i,j) \in E} \left( (y_i^2 - y_j^2) \mathbb{I}\{y_j \leq y_i\} + (y_j^2 - y_i^2) \mathbb{I}\{y_i \leq y_j\}\right) \]
\[ = \sum_{(i,j) \in E} |y_i^2 - y_j^2| \]
\[ = \sum_{(i,j) \in E} |y_i - y_j|(y_i + y_j) \]
\[ \leq \sqrt{\sum_{(i,j) \in E} (y_i - y_j)^2} \sqrt{\sum_{(i,j) \in E} (y_i + y_j)^2} \]

where we applied the Cauchy-Schwartz inequality \( u^\top v \leq \|u\| \|v\| \) in the last step. Using now the elementary inequality \((a + b)^2 \leq 2(a^2 + b^2)\) we may write
\[ \sum_{(i,j) \in E} (y_i + y_j)^2 \leq 2 \sum_{(i,j) \in E} (y_i^2 + y_j^2) = 2d \sum_{i=1}^n y_i^2 \]

Combining the above with (5) we obtain
\[ \frac{\mathbb{E}\left[ |E(S_T, \neg S_T)|\right]}{d \mathbb{E}[|S_T|]} \leq \frac{\sqrt{\sum_{(i,j) \in E} (y_i - y_j)^2}}{d \sum_{i=1}^n y_i^2} \]
\[ = \sqrt{\frac{2 \sum_{(i,j) \in E} (y_i - y_j)^2}{d \sum_{i=1}^n y_i^2}} \]

concluding the proof. \( \square \)

We can now prove Theorem 2.

**Proof of Theorem 2.** Let \( x \in \mathbb{R}^n \) be such that \( x^\top 1 = 0 \) and let \((S_F, \neg S_F)\) be the cut found by Fiedler’s algorithm on input \( x \). Lemma 3 states that:

1. there exists a nonnegative vector \( y \) such that \( R_L(y) \leq R_L(x) \);

2. for this \( y \) and for any \( 0 < t \leq \max_{v \in V} y_v \), the set \( S_t = \{v \in V : y_v \geq t\} \) has at most \( \frac{n}{2} \) vertices (because \( y \) has at most \( \frac{n}{2} \) nonzero components, as we defined it using the median) implying \( \phi(S_t) = \text{xpn}(S_t) \);

3. the cut \((S_t, \neg S_t)\) is one of the cuts considered by Fiedler’s algorithm on input \( x \), which implies \( \phi(S_F) \leq \phi(S_t) \) for all \( t \).

Then, Lemma 5 ensures there exists a threshold \( 0 < t \leq \max_{v \in V} y_v \) such that \( \text{xpn}(S_t) \leq \sqrt{2R_L(y)} \). We can thus write
\[ \phi(S_F) \leq \phi(S_t) = \text{xpn}(S_t) \leq \sqrt{2R_L(y)} \leq \sqrt{2R_L(x)} \]

concluding the proof. \( \square \)
**Nonregular graphs.** What is the correct generalization of the Laplacian matrix $I - \frac{1}{d}A$ when $G$ is not $d$-regular? As we want to preserve the spectral properties, we look at the Rayleigh quotient for the $d$-regular case:

$$\frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{d\sum_{i=1}^nx_i^2}$$

The natural generalization to nonregular graphs is then

$$\frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_{i=1}^nd(i)x_i^2} = \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_{i=1}^n(\sqrt{d(i)}x_i)(\sqrt{d(i)}x_i)} = \frac{x^\top(D - A)x}{(D^{1/2}x)^\top(D^{1/2}x)}$$

where where $D^{1/2} = \text{diag}(\sqrt{d(1)}, \ldots, \sqrt{d(n)})$ and $d(i)$ is the degree of $i$. If we now set $u = D^{1/2}x$, the above becomes

$$\frac{(D^{-1/2}u)^\top(D - A)(D^{-1/2}u)}{u^\top u} = \frac{u^\top D^{-1/2}(D - A)D^{-1/2}u}{u^\top u} = \frac{u^\top(I - D^{-1/2}AD^{-1/2})u}{u^\top u}$$

where we assumed $d(v) > 0$ for all $v$ (there are no isolated vertices) and used $D^{-1/2}DD^{-1/2} = I$.

The matrix $L_{\text{norm}} = I - D^{-1/2}AD^{-1/2}$ whose components are

$$L_{\text{norm}}(i, j) = \begin{cases} \frac{1}{A(i,j)/\sqrt{d(i)d(j)}} & \text{if } i = j \\ -A(i,j)/\sqrt{d(i)d(j)} & \text{otherwise} \end{cases}$$

is known as the normalized Laplacian. All the spectral properties which we proved for $d$-regular graphs, including Cheeger’s inequalities, continue to hold for the normalized Laplacian of arbitrary graphs.

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